Relaxation of stresses in crazes at crack tips and rate of craze extension

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The Barenblatt theory of cohesive stresses at crack tips is used to investigate the effect of the relaxation of craze stresses at crack tips on the rate of craze extension. The craze stresses are equated to the cohesive stresses of the Barenblatt theory. The cancellation by the cohesive/craze stress of the singularity that would exist at the crack tip in their absence is assumed to hold for an extending craze. With this assumption, relaxation of the craze stresses produces craze extension, an effect which has been called 'relaxation controlled growth' by Williams and Marshall. A general equation relating the rate of change of craze length to the rate of change of stress intensity factor (K_I) and the rate of change of the craze stress is derived. It is argued from this equation that uniform crack growth with a constant craze length can occur only at constant K_I. Using plausibility arguments for the behaviour of the craze stress with time and position in the craze, and assuming a generalized Dugdale model, differential equations for the rate of craze extension with no crack growth are derived for the constant load and constant K_I cases. These equations relate the rate of change of craze length to the craze stress at the tip of the crack. Assuming a specific form for the time dependence of this stress, the equation for the constant K_I case is solved to yield an expression for the craze length as a function of time.

Keywords Relaxation; stresses; crazes; crack tips; craze extension; cohesive stresses

INTRODUCTION

At the tips of cracks in most glassy and semi-crystalline polymers under stress, there exists a craze[1]. Under the influence of stress, with the chemical nature of the environment being an important contributing factor, the craze may increase in length (i.e., extend), the crack may grow in length, or both. In this paper we will be concerned with the effect of the viscoelastic nature of the craze material on the rate of craze extension and of crack growth. We have in mind a situation in which a body containing a pre-existing crack is subjected to an external load and a small craze is formed at the tip of the crack. The load may or may not be sufficient to cause crack growth. We ask now the viscoelastic nature of the craze material influences the rate of craze extension. We also consider crack growth under the special circumstance in which the crack grows uniformly with constant length of craze at its tip. However, we do not specifically consider the criterion (such as critical crack opening displacement) for crack growth, nor do we consider craze extension in the absence of cracks. These modes of craze extension and crack growth have been termed 'relaxation controlled growth' by Williams and Marshall[2].

Most theories of the effects of viscoelasticity on the growth of cracks[3–8] have not specifically included the viscoelastic properties of the craze, although the viscoelastic nature of the craze material on crack growth rate has been recognized[5,6]. However, possibly because these theoretical treatments have been more concerned with cracks propagating in rubbery polymers[3–6] where crazing is relatively unimportant compared to glassy or crystalline polymers, the viscoelasticity was considered to reside in the continuum outside the craze region. In these treatments, the craze itself serves only to provide a closing stress at the tip of the crack over a region called the failure zone by Schapery[4] and Knauss[6]. This closing stress is considered to be constant with position along the craze[4,7,8] or as having one region within the craze in which the stress changes linearly, plus a region in which it is constant[6]. In all cases, the stress is considered to be constant in time during the short time necessary for the crack to propagate through the craze region at the tip of the growing crack. For the minute crazes or failure zones that exist at the tip of cracks in elastomers this is an adequate approximation.

The viscoelastic nature of the craze material and the fact that this nature can effect both the growth of cracks and the growth of crazes independent of crack growth has been widely recognized[2,4,10,11]. Thus, Kramer[9] and Lauterwasser and Kramer[12] have argued that in the growth of crazes in the presence of crazing liquids, creep of the craze material is important, while under dry conditions, drawing of the craze material from the substrate is important and creep relatively unimportant. Verheulpen–Heymans and Bauwens[10] considering the growth of crazes in the absence of cracks have argued that the craze far from the craze tip undergoes creep at a stress that is constant with position. Using a theoretical analysis based on the formalism of Muskhelishvili[13] and a nonlinear model of the craze material based upon that of Haward and Thackray[14], they were able to derive an expression for the kinetics of craze growth in the absence of cracks that fit their experimental results.

Most important for our purposes is the work of

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Williams and Marshall[2]. Reasoning from the results of the Dugdale model[15], which was developed for yielding in metals and postulates a constant stress in the yielded zone at the crack tip, they recognized that a reduction of this stress (equated to the stress in the craze) would cause craze growth. The reason for this, as we shall later show in detail for the more general Barenblatt model[16], is that this yield (or craze) stress acts as a closure stress at the tip of the crack and hence cancels the stress singularity that would otherwise exist there. The magnitude of the stress determines the length of the yielded (or crazed) zone, the higher the stress the shorter the zone. Hence, as the stress in a newly formed craze at the tip of a stationary crack decreases due to relaxation effects, the craze zone increases in length in order to cancel the stress singularity. Similarly, for a crack moving with uniform velocity and with a craze zone of constant length at its tip, the velocity of the crack for a given applied stress intensity factor is determined by the relaxation of the stress in the craze from the tip of the craze to its trailing edge (i.e., the tip of the crack). Williams and Marshall[2] termed this 'relaxation controlled growth'. By using the results of the Dugdale model[15] but without rederiving the results for the time dependent case, and by assuming that the stress in the craze zone would decrease in time in the same manner as the relaxation modulus of the uncrazed material, Williams and Marshall were able to calculate average values of the craze stress. From this they developed craze and crack growth laws that showed impressive agreement with experiments.

It is the purpose of this paper to derive more general expressions for the rate of craze and crack growth than were derived by Williams and Marshall[2], but using their concept of relaxation controlled growth. We shall derive these general expressions by starting with Barenblatt's[16] theory of cohesive stresses at crack tips. We then apply these results with a specific empirical expression for the craze stress as a function of time.

DISCUSSION

The model

In our model we shall take the craze to be a viscoelastic material in an elastic continuum. The experiments of Kambour and Kopp[17] and Hoare and Hull[18] show that relative to unyielded polymer, the craze material appears to show greater time dependence. It should be noted that Williams and Marshall[2] considered the continuum to be viscoelastic as well, but since we shall be interested in developing a more rigorous derivation than theirs, taking into account the viscoelasticity of the continuum would add a complexity that would obscure the main lines of the argument.

As our fundamental basis we shall use the Barenblatt[16] theory of cohesive stresses, identifying the craze stress with Barenblatt's cohesive stress. Hence, by the term 'craze stress' we mean the stress provided by the craze on the surface of a crack at its tip. Because we are using the Barenblatt formulation, we can only be concerned with small crazes at the tip of cracks, in which the craze length is very small with respect to the crack length and the dimensions of the specimen. Hence our treatment is valid only for small-scale yielding.

Because we will need the results we will begin with a brief review of the Barenblatt theory. More detailed reviews are given by Goodier[19], Bilby and Eshelby[20], and Schapery[4] whose notation we largely follow.

A diagram of the crack tip situation is shown in *Figure 1*. We consider a centre crack in an infinite medium. The origin of coordinates is at the centre of the crack. At the tip of the crack is the craze. The length of the crack is $2c$ and the distance from the origin to the tip of the craze is a. With respect to an origin at the tip of the craze, we follow Schapery[4] and use coordinates ξ and ξ_1, as shown, ξ denoting positions within the craze/crack and ξ_1 in the uncrazed continuum. The length of the craze is α, which is a function of time, and $\alpha = a - c$. The craze provides a closing stress $\sigma(\xi,t)$ on the surfaces of the craze zone. The surfaces of the crack are stress free. We shall only be concerned with stresses along the axis of crack/craze, i.e., $y=0$. The external loads provide a stress σ_∞ at infinity in the y direction.

It should be noted that Barenblatt does not make the distinction made here between crack and craze. In his formulation, the crack extends to what we have called the craze tip, and his cohesive stress, which we equate with the craze stress, serves to close the tip of the crack. In the absence of a cohesive/craze stress, the crack is imagined to extend to position $x=a$, i.e. the tip of craze, and a stress singularity exists at this point. This definition problem was recognized by Schapery[4] who called what we have called the 'craze tip' the 'crack tip'. In our subsequent discussion we shall follow *Figure 1*.

Barenblatt's approach to the problem is to calculate the stresses and displacements for a crack subject to external loads alone, i.e., without the cohesive stresses. Then a calculation of the stresses and displacements is carried out for a crack subject to the cohesive stresses alone, i.e., without the external loads. The required solution is then the sum of the solution to these two sets of boundary conditions. The more detailed calculation carried out by Goodier[19] involving the sum of solutions to three boundary value problems is unnecessary for our purposes, and we shall only give the results necessary to the subsequent exposition.

For the crack without cohesive stresses, the results for the stresses σ_x^0 and σ_y^0 and displacements v^0 along the x axis ($y=0$) but very near the crack tip, are

$$\sigma_y^0 = \sigma_x^0 = \frac{K_1}{(2\pi\xi_1)^{1/2}}, \quad \xi_1 > 0 \tag{1}$$

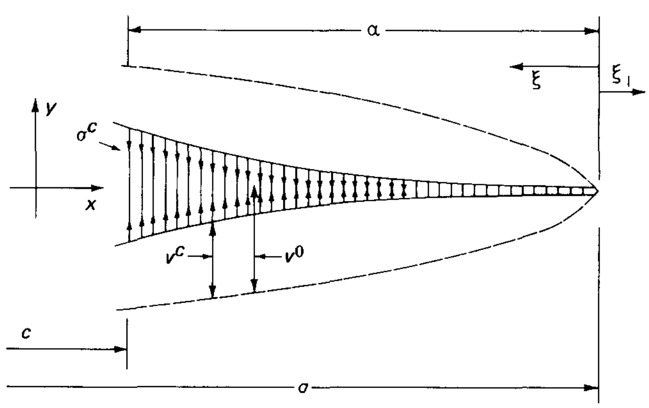

Figure 1 Schematic representation of the configuration at the crack tip. The crack extends to the position *c*. The region between *c* and *a* is occupied by the craze which provides closing stresses as shown. The broken line represents the displacement in the absence of the craze stresses

$$v^0 = C_e[K_1/(2\pi)^{1/2}]\xi^{1/2}, \quad \xi > 0 \tag{2}$$

In these equations, K_1 is the stress intensity factor for mode I opening, and is a function of the external applied stresses and the geometry of the situation. It represents the action of the external stress. The quantity C_e is the compliance in plane strain and is given by[4]

$$C_e = 4(1 - v^2)/E \tag{3}$$

where v is Poisson's ratio and E is the Young's modulus of the material in the continuum (not the craze material).

The cohesive/craze stresses σ, acting alone in the region α cause stresses σ_x^c and σ_y^c to the right of the craze tip and displacements v^c to the left of it given by

$$\sigma_x^c = \sigma_y^c = -\frac{1}{\pi \xi_1^{\;1/2}} \int_0^\alpha \sigma_c(\xi,t) \frac{\xi^{1/2} d\xi}{(\xi + \xi_1)}, \quad \xi_1 > 0 \tag{4}$$

$$v^c(\xi) = -\frac{C_e}{2\pi} \int_0^\alpha \sigma_c(\zeta,t) \ln \left| \frac{\zeta^{1/2} + \xi^{1/2}}{\zeta^{1/2} - \xi^{1/2}} \right| d\zeta, \quad \xi > 0 \tag{5}$$

where $\sigma_c(\zeta,t)$ is the distribution of cohesive/craze stresses which we have specifically indicated to be a function of time as well as position in the zone α where they act, and ζ is a variable of integration. Very near the crack tip, i.e., $\xi_1/\alpha \ll 1$, equation (4) becomes[4]

$$\sigma_x^c = \sigma_y^c = -\frac{1}{\pi(\xi_1)^{1/2}} \int_0^\alpha \frac{\sigma_c(\xi,t)}{\xi^{1/2}} d\xi + \sigma_c(0), \quad \xi_1 > 0 \tag{6}$$

The stresses in the continuum to the right of the tip of the craze under the combined action of the external stresses and the cohesive stresses are obtained by adding equations (1) and (6) (or (4)). A comparison of equations (1) and (6) shows that the resulting stresses in the continuum will be finite only if

$$K_1 = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\alpha \frac{\sigma_c(\xi,t)}{\xi^{1/2}} d\xi \tag{7}$$

As pointed out by Schapery, a more complete representation is obtained by adding equations (1) and (4) and using equation (7) to obtain

$$\sigma_x = \sigma_y = \frac{\xi_1^{1/2}}{\pi} \int_0^\alpha \frac{\sigma_c(\xi,t) d\xi}{\xi^{1/2}(\xi + \xi_1)}, \quad \xi_1 > 0 \tag{8}$$

The displacements are obtained by adding equations (2) and (5) and using equation (7). This gives

$$v(\xi) = \frac{C_e}{2\pi} \int_0^\alpha \sigma_c(\zeta) \left\{ 2(\xi/\zeta)^{1/2} - \ln \left| \frac{\zeta^{1/2} + \xi^{1/2}}{\zeta^{1/2} - \xi^{1/2}} \right| \right\} d\zeta, \quad \xi \geq 0 \tag{9}$$

Equations (7), (8), and (9) (and their viscoelastic analogues) are the equations that relate the stresses and displacements to the external stresses, the cohesive stresses, and the stress–strain properties of the elastic continuum. The stress–strain properties of the material providing the cohesive stresses nowhere appear explicitly. The stresses provided by it serve only to close the edges of the crack (equation (5)) from where they would have been in their absence (equation (2)). This point is important because we identify the material in the cohesive zone with the craze and we will later discuss the effects of the viscoelastic properties of the craze material.

For our purposes, the important equation is equation (7). This equation expresses the relationship between $\sigma_c(\xi,t)$ and α such that the singularity at the craze tip be cancelled. The higher $\sigma_c(\xi,t)$, the shorter α and conversely. Thus, if after the formation of the craze the stress in it were to decrease by relaxation, then the length α must increase in order to maintain the absence of a stress singularity at the craze tip.

An inspection of the displacement equations (2), (5), and (9), however, indicates that the craze material is not subject to either a simple stress–relaxation (constant elongation) or a simple creep (constant stress) situation. At any point x in the craze, as the stress in the craze decreases, the surfaces of the craze at that point will begin moving toward their stress-free position as given by equation (2). This in turn will provide more elongation for the craze material at that point, tending to retard the decrease of stress. The situation is analogous to carrying out a stress relaxation experiment on a 'soft' machine, i.e., one in which the displacement increases as the load decreases. As pointed out by Barenblatt[16] and Bilby and Eshelby[20] for the elastic case, a full formulation of this problem leads to intractable integral equations; viscoelasticity further complicates the matter. For the present, it seems reasonable to conclude that because of the nature of the mechanical environment in which it finds itself, the rate of relaxation of the stress within the craze will be slower than it would be in a pure stress relaxation experiment.

Growth of the craze

We may now use equation (7) to develop an expression for the rate of growth of the craze at the tip of a crack. We shall not be concerned with any microscopic mechanisms of craze initiation as formulated by Argon[21], Gent[22], and others. Rather, we assume that at some time $t = 0$ a craze has been established at the crack tip, and we ask ourselves how the time dependence of the craze stress determines the rate of growth of the craze. This model implies that the criterion for craze growth is a stress criterion. At the tip of the initial crack (before craze growth) and at the tip of the craze (during subsequent growth) the stress σ_y must reach some value σ_0 in order for the craze to grow. Because of the triaxial nature of the stress under the plane-strain conditions at these points, this seems a reasonable implicit assumption. Also, because of this, no induction time is predicted, which again seems to be reasonable for crazes at crack tips. Differentiating equation (7) with respect to time we obtain the craze/crack growth law for relaxation controlled growth:

$$\frac{\sigma(\alpha,t)}{\alpha^{1/2}} \frac{d\alpha}{dt} = \sqrt{\frac{\pi}{2}} \frac{dK_I}{dt} - \int_0^\alpha \frac{\partial \sigma(\xi,t)}{\partial t} \frac{d\xi}{\xi^{1/2}} \tag{10}$$

where by $\sigma(\alpha,t)$ we mean the stress at time t at the trailing edge of the craze, namely the tip of the crack.

Equation (10) is the basic equation governing craze growth resulting from the imposition of the Barenblatt condition that the craze stress cancels the singularity that would otherwise occur at the craze tip. For a crazing stress that is constant in time the equation gives the expected result that the craze grows only if K_I is changed. For a constant K_I, the craze will grow at a rate dependent on the length of the craze, the stress on the craze at the trailing edge, and the integral over the crazed region of the rate of change of the craze stress. Since this is expected generally to be negative, the equation predicts that growth rate will be positive. On the basis of this model, the craze will not grow at constant K_I if the craze stress is independent of time, a point which has been made by Kramer[9].

In this latter respect, we first examine the situation in which the crack advances with uniform velocity and with constant craze length, i.e., $\dot{c}=\dot{a}=u>0$, where u is the constant velocity. As has been pointed out by McCartney[7], under these conditions, the displacement, and hence the stress, will be a function only of $x-ut$. But since under these conditions $\xi=ut-x$, the derivative (which is at constant ξ) within the integral in equation (12) is zero, and hence crack propagation at constant velocity and at constant craze length can only occur if K_I is a constant, as has already been pointed out by McCartney[7] in a different context. The length of the craze will depend upon the value of K_I through equation (7), as will the crack velocity and the value of the craze stress, but without assuming some criterion such as critical crack opening displacement[2] for crack advance, not much more can be said about the relationship between K_I and crack velocity from this formulation.

Turning now to the problem of craze extension without crack growth (i.e., $\dot{c}=0$, $\dot{a}\neq 0$) we note from equation (10) that the actual calculation of craze extension requires the knowledge of the time dependence of the craze stress for all points within the craze for the general loading situation. As already mentioned, this is a formidable problem which can be formulated in terms of intractable integral equations. However, a reasonable approach may be attempted using some physical reasoning. To pursue this, following Schapery[4] let us first transform equation (12) by the substitution $\eta=\xi/\alpha$, which normalizes the craze length. Then we consider two separate situations: constant load and constant K_I, although we will mainly be interested in the latter case. For a constant load providing a stress σ_∞ in the y direction at infinity, from the Barenblatt theory we have for the centre-crack case considered here:

$$K_I=\sigma_\infty\sqrt{\pi a} \qquad (11)$$

We then obtain from equation (10):

$$\frac{\sigma(\alpha,t)}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}\left[1-\frac{\pi}{2\sqrt{2}}\frac{\sigma_\infty}{\sigma(\alpha,t)}\sqrt{\frac{\alpha}{c+\alpha}}\right]=-\int_0^1\frac{\partial\sigma}{\partial t}\frac{\mathrm{d}\eta}{\eta^{1/2}} \qquad (12)$$

For the case of constant K_I, we obtain

$$\frac{\sigma(\alpha,t)}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}=-\int_0^1\frac{\partial\sigma}{\partial t}\frac{\mathrm{d}\eta}{\eta^{1/2}} \qquad (13)$$

It will be recognized that equation (12) reduces to equation (13) if:

$$\frac{\pi}{2\sqrt{2}}\frac{\sigma_\infty}{\sigma(\alpha,t)}\sqrt{\frac{\alpha}{c+\alpha}}\ll 1 \qquad (14)$$

The Barenblatt theory is applicable when $\alpha/c\ll 1$, and since $\sigma(\alpha,t)$ is not expected to be a great deal smaller than σ_∞, we shall henceforth consider the constant K_I case only.

We now consider qualitatively the dependence of the craze stress as a function of time and η as the craze extends. Similar considerations have already been used by Wnuk[8] for the dependence of the craze stress on relaxation in the surrounding continuum. The situation is depicted schematically in *Figure 2a*. By hypothesis, the stress at the leading edge of the craze at all times is the value of the craze initiation stress σ_0. At time $t=0$ we assume some stress distribution which may be a constant or similar to any of those determined by Lauterwasser and Kramer[12]. As time proceeds, those points furthest removed from the craze tip will be 'older' and hence the

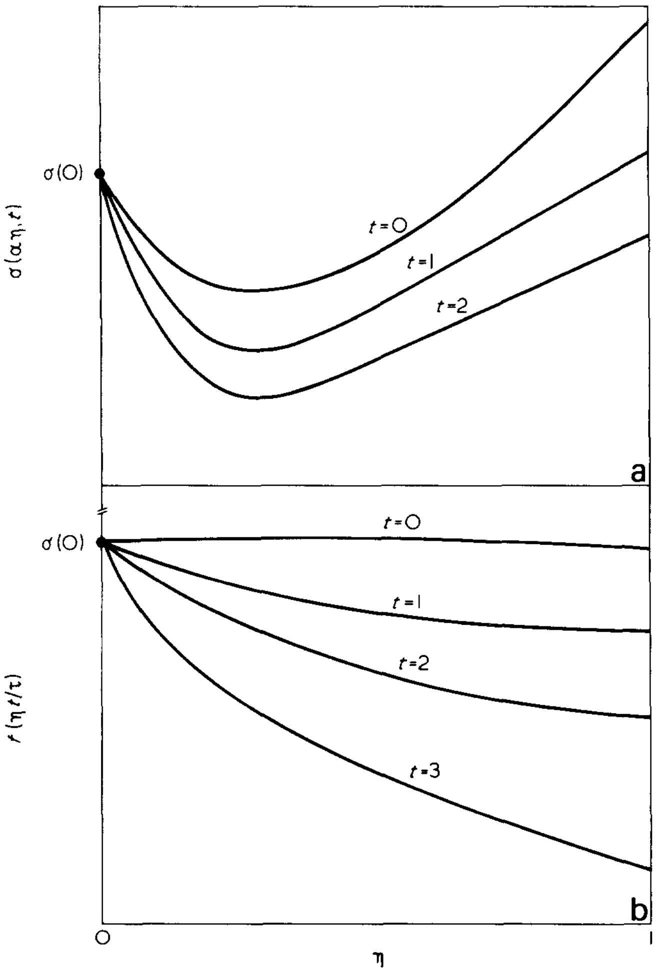

Figure 2 (a) Schematic representation of the behaviour with time over the normalized craze length of the stresses in extending craze. By hypothesis the stress at the craze tip is a constant. (b) Same as (a) but with the craze stress at t = 0 constant with position. This is a generalized Dugdale model

stress will be lower, and in *Figure 2a* we have attempted to show this schematically for times greater than $t=0$. Without knowledge of the relationship between η and t equation (13) cannot be solved analytically, although it could be solved numerically if the time dependence of the stress were known.

In view of the above discussion, let us make the simplest approximation possible for the form of $\sigma(\alpha\eta,t)$. Let us assume that:

$$\sigma(\alpha\eta,t)=f(\eta t/\tau) \tag{15}$$

in which τ is some normalizing parameter in the nature of a relaxation time, but should not be confused with the relaxation time for the relaxation modulus of the material. Clearly we must have:

$$\sigma_0=f(0) \tag{16}$$

and hence, for $t=0$, $\sigma(\alpha\eta,0)$ is a constant independent of η. This, then, is the Dugdale model[15] with relaxation, and is shown schematically in *Figure 2b*. At the tip of the craze ($\eta=0$) the stress is always σ_0. At the trailing edge of the craze ($\eta=1$), the stress is given by:

$$\sigma(\alpha,t)=f(t/\tau) \tag{17}$$

With these definitions we may now evaluate the integral in equation (13). Denoting the integral by I, we substitute equation (15) into it and obtain:

$$I=\frac{1}{\tau}\int_0^1 f'\eta^{1/2}\mathrm{d}\eta \tag{18}$$

where the prime denotes differentiation with respect to the argument. This may now be integrated by parts to obtain:

$$I=\frac{1}{\tau}\left(\frac{\tau}{t}\right)^{3/2}\left[\left(\frac{t}{\tau}\right)^{1/2}f(t/\tau)-\frac{1}{2}\left(\frac{t}{\tau}\right)^{1/2}\int_0^1\frac{f(\eta t/\tau)}{\eta^{1/2}}\mathrm{d}\eta\right] \tag{19}$$

Now, it can be seen immediately from the definition of η and equation (7) that the integral on the RHS of equation (19) is equal to $\sqrt{\frac{\pi}{2}}K_I/\alpha^{1/2}$. Substituting into equation (19) we obtain:

$$I=\frac{1}{t}\left[f(t/\tau)-\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}K_I/\alpha^{1/2}\right] \tag{20}$$

This may now be substituted into equation (13) to give:

$$\frac{\sigma(\alpha,t)}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\frac{1}{t}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}\frac{K_I}{\alpha^{1/2}}-\sigma(\alpha,t)\right] \tag{21}$$

which, upon collecting terms, gives a final differential equation for the rate of growth of the craze:

$$\frac{1}{\alpha^{1/2}}\frac{\mathrm{d}\alpha}{\mathrm{d}t}+\frac{1}{t}\alpha^{1/2}=\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{K_I}{t\sigma(\alpha,t)} \tag{22}$$

The analogous equation for the constant load case is:

$$\frac{1}{\alpha^{1/2}}\frac{\mathrm{d}\alpha}{\mathrm{d}t}\left[1-\frac{\pi}{2\sqrt{2}}\frac{\sigma_\infty}{\sigma(\alpha,t)}\sqrt{\frac{\alpha}{c+\alpha}}\right]+\frac{\alpha^{1/2}}{t}$$
$$=\frac{1}{t}\frac{\pi}{2\sqrt{2}}\frac{\sigma_\infty}{\sigma(\alpha,t)}\sqrt{c+\alpha} \tag{23}$$

While equation (23) is a nonlinear equation that cannot be integrated easily, equation (22) is linear and can be integrated immediately to give:

$$\alpha^{1/2}t^{1/2}=\frac{1}{4}\sqrt{\frac{\pi}{2}}K_I\int_0^t\frac{\mathrm{d}t}{t^{1/2}\sigma(\alpha,t)} \tag{24}$$

This is our equation for relaxation controlled growth at constant K_I. It involves three basic assumptions: (1) the stress in the craze acts like the Barenblatt cohesive stresses to cancel the singularity at the tip of the craze, (2) the criterion for crazing is a value σ_0 of the craze stress at the craze tip, and (3) the craze stress behaves within the craze according to equation (15). The last of these is considered to be the most drastic of the assumptions. It is applicable only so long as the craze is very short in comparison to the crack length.

To go beyond this we need to know the behaviour with time of the stress of the trailing edge of the craze. Note that so far we have said nothing about the mechanical behaviour of the craze material. In particular, the material need not be a linear viscoelastic material, but can be nonlinear as well. To proceed further, we now assume that the stress behaves in a manner similar to a linear viscoelastic material. To the knowledge of this author, aside from the determination of stress–strain curves by Kambour and Kopp[17] and Hoare and Hull[18], no data have been published on the viscoelastic nature of the craze material. Comparing the values of craze stress obtained by various authors[23–26] with the modulus figures given by Kambour and Kopp and Kramer[9], elongations of two to ten per cent would be obtained for the craze material, if linear. At the lower range of stresses the material might be considered linear, at the upper ranges it is clearly not. Kambour and Kopp have further shown that on the first cycle of a stress–strain curve, ~55% of the strain is nonrecoverable, and subsequent stress–strain cycles show curves that are not drastically nonlinear. Thus there is some reason to hope that, at least for low craze stresses, linear viscoelasticity may be approximately applicable, and that the stress-bearing extension of the craze may be only a small fraction of the total displacement.

With these remarks we follow Williams and Marshall[2] and assume a power law for the relaxation of the stress in the craze material. As previously described, because of the mechanical environment of the craze there is no *a priori* justification for assuming that the exponent in this power law is the same as that of the relaxation modulus. Nevertheless, since we wish to have a stress that is finite at $t=0$, we assume for the stress:

$$\sigma(\alpha,t)=\frac{\sigma_0}{1+(t/\tau)^m} \tag{25}$$

which is essentially the reciprocal of the compliance function used by Schapery[4]. For times long compared to τ, this equation reduces to the commonly used power law for relaxation modulus[2].

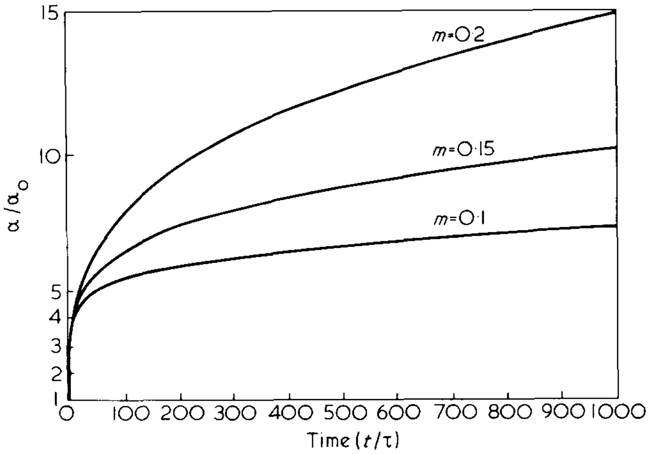

Figure 3 The behaviour of the craze length with time at the tip of a stationary crack according to equation (26). The craze stress at the tip of the crack is assumed to behave according to equation (25)

This may be substituted into equation (24) and integrated immediately to give:

$$\alpha = \frac{\pi}{8}\frac{K^2}{\sigma_0^2}\left[1+\left(\frac{t}{\tau}\right)^m \frac{1}{2m+1}\right]^2 \qquad (26)$$

At time $t=0$ the initial length of the craze is $\frac{\pi}{8}K^2/\sigma_0{}^2$ which is the expected result for a constant stress over the craze zone as in the Dugdale model for small-scale yielding. As time increases, the craze extends at a decreasing rate. A plot is given in *Figure 3* for three values of m. For $m=1/2$ the curves become linear in t for values of time long with respect to τ.

Except for the fact that these curves do not start at zero length (the initial craze length may be difficult to determine experimentally, see later) these curves show a general resemblance to the results of Marshall, Culver, and Williams[27] and Narisawa and Kondo[28]. For comparison with the results of this theory, both these investigations were complicated by the presence of solvent. The former authors measured craze growth in PMMA in methanol. They ascribed the time dependence of their results to the flow of methanol through the craze, an explanation which was supported by the dependence of the craze length on the square root of the time for the early portions of craze growth. The latter authors investigated craze growth in both PMMA and polycarbonate with craze initiation and growth hastened by kerosene. In PMMA they found a square root dependence on time, but in polycarbonate the dependence was on the 0.24 power of the time, a result which is difficult to ascribe to solvent diffusion effects. However, these authors also found a linear dependence of craze length on K in contrast to the results presented here. The K dependence of the Marshall, Culver, and Williams results appears to be somewhere between the first and second power. These results indicate that in the presence of solvent, flow of the solvent through the craze is the principal rate controlling process, but relaxation of the craze stress may also have some bearing on the rate of craze extension.

For times long compared to τ, the length of the craze is given approximately by:

$$\alpha \cong \frac{1}{4(m+\frac{1}{2})^2}\frac{\pi}{8}\frac{K^2}{\sigma_0{}^2}\left(\frac{t}{\tau}\right)^{2m} \qquad (27)$$

which, except for the numerical factor and the presence of τ, is identical to the expression developed by Williams and Marshall[2] by less rigorous means, and used by them to analyse the growth kinetics of crazes at crack tips in the absence of solvent in a number of polymers with impressive success.

CONCLUSION

Starting with the Barenblatt solution for the stresses at the tip of a crack which is closed by the cohesive stresses at the crack tip, and assuming that the Barenblatt hypothesis that the cohesive stresses cancel the singularity at the craze tip during crack and craze extension, a general expression (equation (10)) has been derived for the rate of change of craze length. The craze stress is equated to the cohesive stress, as was done by Schapery[4] and craze extension is caused by the relaxation of the stress in the craze. Thus, equation (10) relates the rate of change of craze length to the integral of the time rate of change of craze stress in the craze zone. The equation is applicable only so long as the craze is very short compared to the crack length. This mode of craze growth was termed 'relaxation controlled growth' by Williams and Marshall[2] but no general expression for the growth was given.

It is argued that the mechanical situation for the craze at a crack tip corresponds to neither pure creep (constant stress) nor pure relaxation (constant extension), and hence the rate of change of craze stress is not easily relatable to the relaxation or creep functions for the craze material. Nevertheless, a physically plausible form (equation (15)) for the dependence of craze stress on location within the craze and time is hypothesized. Using this expression, equation (10) is solved for constant load and for constant stress intensity factor, and differential equations (equations (22) and (23)) for the length of the craze as a function of time are derived. The differential equation for constant K_I is solved to give an equation (equation (24)) for the length of the craze as an integral over time of the craze stress at the trailing edge of the craze (the tip of the crack). Then, giving a qualitative justification for the use of linear viscoelasticity, and using a power law similar to that previously used[2,5] a specific expression for the craze length at a crack tip is derived. The form of this expression shows a similarity to the results of craze growth experiments in PMMA in methanol[27] and kerosene[28] and polycarbonate in kerosene[28]. While for comparison with the present results these experiments are complicated by the need to take into account solvent flow, the comparison indicates that relaxation controlled growth may be a contributing factor. At longer times, the expression reduces (within a numerical factor) to one previously derived by less rigorous means[2] and shown to agree with experiment very well[2].

In this formulation, no induction time is predicted. Since the concern here is with crazes at crack tips, an induction time is not expected. The predictions given here for the time dependence of the length of crazes at crack tips are widely different from the familiar linear increase of length with the logarithm of time[10,11] which occurs in the absence of macroscopic cracks. However, even in that case the viscoelastic nature of the craze material is important in determining the kinetics of craze growth[10].

A more serious deficiency is the lack of a threshold stress intensity factor before crazing begins, as observed by Marshall, Culver, and Williams[27], Narisawa and Kondo[28], and Israel, Thomas, and Gerberich[29]. Given the presence of a microscopically sharp crack, it seems implicit from a stress criterion for crazing that no threshold value of K would exist. If a microscopically sharp crack can be hypothesized in these experiments, then the presence of a threshold value of K implies a minimum length of craze. That is, a craze, being a collection of fibrils and voids, cannot exist as a separate 'phase' below a given size. From the data in ref. 29 this minimum length appears to be of the order of 5 micrometres. Clearly, the theory presented in this paper has not addressed the problem of the minimum length of craze, or what is equivalent, the existence of a threshold stress intensity. We have attempted to answer the far simpler question of, given a craze at a crack tip, how is its growth determined by the relaxation of the stress in the craze? In answering this question we believe we have put the work of Williams and Marshall[2], who first addressed this question, on a somewhat firmer footing.

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REFERENCES

1 Kausch, H. H. 'Developments in Polymer Fracture', Springer-Verlag, Berlin (1978) p. 274
2 Williams, J. G. and Marshall, G. P. *Proc. Roy. Soc. (London)* 1975, **A342**, 55
3 Williams, M. L. *Int. J. Fract. Mech.* 1965, **1**, 292
4 Schapery, R. A. *Int. J. Fract.* 1975, **11**, 141
5 Schapery, R. A. *Int. J. Fract.* 1975, **11**, 369
6 Knauss, W. G. in 'Deformation and Fracture of High Polymers', (Eds. H. H. Kausch, J. A. Hassell and R. I. Jaffee), Plenum Press, New York, (1973), pp. 501–541
7 McCartney, L. N. *Int. J. Fract.* 1977, **13**, 641
8 Wnuk, M. G. *J. Appl. Mech.* 1974, **41**, 234
9 Kramer, E. J. in 'Developments in Polymer Fracture—1', (Ed. E. H. Andrews), Applied Science Publ., Ltd., London (1979), pp. 55–120
10 Verheulpen-Heymans, N. and Bauwens, J. C. *J. Mater. Sci.* 1976, **11**, 1
11 Verheulpen-Heymans, N. and Bauwens, J. C. *J. Mater. Sci.* 1976, **11**, 7
12 Lauterwasser, B. J. and Kramer, E. J. *Phil. Mag.* 1979, **A39**, 469
13 Muskhelishvili, N. I. 'Some Basic Problems of the Mathematical Theory of Elasticity', P. Noordhoff, Groningen, 1953
14 Haward, R. N. and Thackray, G. *Proc. Roy. Soc.* 1968, **302I**, 453
15 Dugdale, D. S. *J. Mech. Phys. Solids* 1960, **8**, 100
16 Barenblatt, G. I. 'The Mathematical Theory of Equilibrium Cracks in Brittle Fracture', Adv. in Appl. Mech., Vol. VII, Academic Press, New York, 1969, pp 55–129
17 Kambour, R. P. and Kopp, R. W. *J. Polym. Sci., Part A2* 1969, **7**, 183
18 Hoare, J. and Hull, D. *Phil. Mag.* 1976, **26**, 443
19 Goodier, J. N. 'Fracture, an Advanced Treatise', Vol. II, (Ed. H. Liebowitz), Academic Press, New York, 1968
20 Bilby, B. A. and Eshelby, J. D. in 'Fracture, an Advanced Treatise', Vol. 1, (Ed. H. Liebowitz), Academic Press, New York, 1968
21 Argon, A. S. and Hannoosh, J. G. *Phil. Mag.* 1977, **36**, 1196
22 Gent, A. N. *J. Macromol. Soc., Phys.* 1973, **B8**, 597
23 Brown, H. R. and Ward, I. M. *Polymer* 1973, **14**, 469
24 Morgan, G. P. and Ward, I. M. *Polymer* 1977, **18**, 87
25 Pitman, G. L. and Ward, I. M. *Polymer* 1979, **20**, 895
26 Weidmann, G. W. and Döll, W. *Non-Crystalline Solids* 1977, 606
27 Marshall, G. F., Culver, L. E. and Williams, J. G. *Proc. Roy. Soc., London* 1970, **A319I**, 165
28 Narisawa, I. and Kondo, T. *J. Polym. Sci., Polym. Phys. Edn.* 1973, **11**, 223
29 Israel, S. J., Thomas, E. L. and Gerberich, W. W. *J. Mater. Sci.* 1979, **14**, 2128